Math 255A' Lecture 23 Notes

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1 Spectral Theorem for Normal Operators

1.1 Spectral theorem for normal operators

Let T be a self-adjoint, bounded operator on a Hilbert space H. We have shown that

$$T = \int_{[a,b]} \lambda \, dE(\lambda),$$

in the sense that

$$\langle Tx, y \rangle = \int_{[a,b]} \lambda \, d\langle E(\lambda), x, y \rangle,$$

where $d\langle E(\lambda), x, y \rangle$ is the Lebesgue-Stieltjes measure given by the map $\lambda \mapsto \langle E(\lambda), x, y \rangle$.

We can extend our functional calculus to Borel-measurable functions by defining $f(T)$ to satisfy

$$\langle f(T)x, y \rangle = \int_{[a,b]} f(\lambda) \, d\langle E(\lambda), x, y \rangle.$$

So we can construct a **spectral measure** $E : \mathcal{B}([a,b]) \to \mathrm{Proj}(H)$ such that

- $E(\varnothing) = 0$, $E([a,b]) = I$,
- If A_n are disjoint, $E(\bigcup_{n=1}^\infty A_n)x = \sum_{n=1}^\infty E(A_n)x$ for all x (this is a weak operator convergent sum).
- $E(A \cap B) = E(A)E(B)$ for all $A, B \in \mathcal{B}([a,b])$.

Diagonalization of an operator looks like

$$T = \sum_{\lambda \in \sigma_p(T)} \lambda P_\lambda.$$

In the self-adjoint case, all λs must be real. In the case of normal operators, λ may be complex.

Theorem 1.1 (Spectral theorem for bounded normal operators)**.** *Let H be a Hilbert space over $\mathbb{C}$, and let $N \in \mathcal{B}(H)$ be normal. Then there is a compact $D \subseteq \mathbb{C}$ and a spectral measure $E : \mathcal{B}(D) \to \operatorname{Proj}(H)$ such that*

$$N = \int_D z \, dE(z).$$

In other words,

$$\langle Nx, y \rangle = \int z \, d\langle E(z)x, y \rangle,$$

where $U \mapsto \langle E(U)x, y \rangle$ is a complex-valued measure for each $x, y \in H$.

Given N, we can write $S + iT$, where S, T are both self-adjoint and commute.

Lemma 1.1. *In the spectral representation of T, every $E(\lambda)$ commutes with every operator that commutes with T.*

Proof. If $p \in \mathbb{R}[t])$ and S commutes with T, then S commutes with $p(T)$. Commutativity survives for $f(T)$ with $f \in C([a,b])$ by convergence in operator norm. Finally, if $T_n \xrightarrow{WO} T$ and $ST_n = T_n S$ for all n, then

$$\langle STx, y \rangle = \langle Tx, S^* y \rangle = \lim_n \langle T_n x, S^* y \rangle = \langle TSx, y \rangle ,$$

for all $x, y \in H$, so $ST = TS$. □

Corollary 1.1. *$E^S(\mu)E^T(\lambda) = E^T(\lambda)E^S(\mu)$ for all λ, μ.*

Proof. Apply the lemma twice. □

Now, given $(p,q] + i(r,s] \subseteq \mathbb{C}$, define

$$\begin{aligned} E((p,q] + i(r,s]) &:= E^S((p,q]) + iE^T((r,s]) \\ &= (E^S(q) - E^S(p)) + i(E^T(s) - E^T(r)). \end{aligned}$$

Warning: We may have $\left\langle E^S((p,q])E^T((r,s])x, u \right\rangle \neq \left\langle E^S((p,q])x, y \right\rangle \left\langle E^T((r,s]x, y \right\rangle$.

Let $a^S = \inf \langle Sx, x \rangle$ and $b^S = \sup \langle Sx, x \rangle$, and define a^T and b^T similarly. We can now define

$$N' = \int_D z \, dE(z), \qquad D = [a^S, b^S] + i[a^T, b^T].$$

We want to check that $N' = N$. Using the spectral theorem for self-adjoint operators,

$$N' = \int_D x \, dE(z) + i \int_D y \, dE(z) = S + iT = N.$$

The middle step is by a "Fubini"-type argument.

1.2 Approximate eigenvalues

In the compact case, we had $T = \sum_{i=1}^{\infty} \lambda_i P_{\lambda_i}$, where the λ_i were the eigenvalues of T.

Definition 1.1. Let X be a normed space. $\lambda \in \mathbb{C}$ is an **approximate eigenvalue** for $T \in \mathcal{B}(X)$ if

$$\inf\{\|(T-\lambda)x\| : \|x\| = 1\} = 0.$$

For compact operators, we saw that these were actually eigenvalues of the operator. In general, this isn't true. Here is an example:

Proposition 1.1. *Let $H = L^2([0,1])$, and let $Tf(x) = xf(x)$. Then λ is an approximate eigenvalue if and only if $\lambda \in [0,1]$.*

Proof. Let $0 \le \lambda \le 1$, so $(T-\lambda)f(x) = (x-\lambda)f(x)$. For any $\varepsilon > 0$, pick $f \in L^2([0,1])$ with $\|f\| = 1$ such that $f(x) = 0$ if $x \notin (\lambda - \varepsilon, \lambda + \varepsilon)$. Then $\|Tf\| \le \varepsilon\|f\|$. □

How does this play into our spectral representation, $T = \int_{[a,b]} \lambda \, dE(\lambda)$?

Definition 1.2. The **support** of E is $\{\lambda : E(\lambda + \varepsilon) \neq E(\lambda - \varepsilon) \, \forall \varepsilon > 0\}$.

Proposition 1.2. *The support of E is the set of approximate eigenvalues for T.*

Proof. ($\supseteq$): Suppose that $E(c) = E(d)$ for some $a \le c < d \le b$, and let $c < \mu < d$; we will show that μ is not an approximate eigenvalue. Then

$$T = \int_{[a,b]} \lambda \, dE(\lambda) = \int_{[a,c]} \lambda \, dE(\lambda) + \int_{[d,b]} \lambda \, dE(\lambda),$$

so we get

$$T - \mu = \int_{[a,b]} \lambda \, dE(\lambda) = \underbrace{\int_{[a,c]} (\lambda - \mu) \, dE(\lambda)}_{T_1} + \underbrace{\int_{[d,b]} (\lambda - \mu) \, dE(\lambda)}_{T_2}.$$

Both T_1 and T_2 are reduced by $I = E(x) + E(d,b]$. If we write $x = x_1 + x_2$, then

$$\|T_1 x_1\| \ge |c - \mu| \|x_1\|, \qquad \|T_2 x_2\| \ge |d - \mu| \|x_2\|,$$

so we cannot make these arbitrarily small.

($\supseteq$): If $\mu \in \operatorname{spt}(E)$, let $\varepsilon > 0$. Then $E(\mu - \varepsilon, \mu + \varepsilon] \neq 0$. Pick x with $\|x\| = 1$ such that $E(\mu - \varepsilon, \mu + \varepsilon]x = x$. Then

$$\langle (T-\mu)x, y \rangle = \int_{[a,b]} (\lambda - \mu) \, dE(\lambda) x = \int_{[\mu - \varepsilon, \mu + \varepsilon} (\lambda - \mu) \, d\langle E(\lambda)x, y \rangle \le \varepsilon \|x\| \|y\|. \quad \square$$

This will give us a better idea of what the set D is. It will be a set of eigenvalues.

1.3 Banach algebras

Definition 1.3. An **algebra** over $\mathbb{F}$ is a vector space A over $\mathbb{R}$ togetehr with a multiplication $A \times A \to A : (a, b) \mapsto ab$ which is associative and distributive with addition. An algebra has an **identity** if there is some $e \in A$ such that $ea = ae = a$ for all $a \in A$.

Definition 1.4. A **Banach algebra** is a Banach space A which is also an algebra such that

$$\|ab\| \leq \|a\|\|b\|, \qquad \forall a, b \in A.$$

Example 1.1. Let X be compact and Hausdorff. Then $C(X)$ is a Banach algebra. If X is locally compact, $C_0(X)$ is a Banach algebra.

Example 1.2. $L^\infty(\mu)$ is a Banach algebra.

These are all commutative. Here are some noncommutative examples.

Example 1.3. $\mathcal{B}(X)$ is a Banach algebra if X is a Banach space.

Example 1.4. The collection of compact operators, $\mathcal{B}(X)$, is a Banach algebra (when X is a Banach space).